THE STRUCTURE OF RECOGNIZABLE DIATONIC TUNINGS
by EASLEY BLACKWOOD

A review by Carlo Serafini[1]

The author introduces this subject explaining the basic properties of musical intervals and observing that "*the structure of recognizable tunings is basically an array of intricate interconnections among acceptable approximations to intervals, certain of which may be tuned individually so as to be free of beats produced by interacting harmonics*".

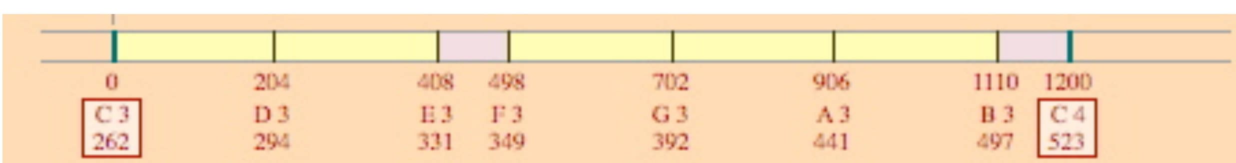
The most important intervals playing a fundamental role in the evolution of scales and tunings are ratios 2:1, 3:2 and 5:4 (octave, just fifth and just major third) and since the octave "*cannot be tuned any other way than in the ratio 2:1*" (here I beg to differ as we will see later on), the adjustment (temperament) of the other two basic intervals is what the history of Western tuning systems is all about.

The fact that building scales using these basic intervals is an intricate matter is immediately clear simply because interval ratios built with powers of prime numbers 2, 3 and 5 never coincide.

Blackwood generally describes interval ratios as combinations of these first three intervals so, for example, a major second equals 2 just fifths minus an octave. Because all intervals are created from the three basic ones, lots of other microintervals (commas) start appearing when we try building scales, due to the above mentioned incompatibility among ratios derived from prime numbers 2, 3 and 5.

The analysis of recognizable diatonic tunings starts with the one based only on the first two basic intervals (2:1 and 3:2), called Pythagorean[2], according to the following definition: "*let us begin with any pitch whatever and find another pitch higher than the first one by the second interval, whose ratio is 3:2*". If we repeat this process until we have seven pitches that are then transposed within a compass of an octave and rearranged in ascending order, we have a diatonic scale.
All intervals are numbered according to the place they have on the "line of fifths" so, the fifth above root note (0) is interval (1) and the one below it is interval (-1).

A Pythagorean major scale has two different interval sizes: the large one appears five times and the small one twice (a characteristic shared by all diatonic tunings):



[1] http://www.seraph.it/

[2] https://en.wikipedia.org/wiki/Pythagorean_tuning

Blackwood calls these two intervals (2) and (-5) because we would find the large one two places above the starting point on the line of fifths and the small one, five places below it. Things clearly get tricky quickly!
For example, the Pythagorean major third is larger than the third basic interval (just major third) and the difference in ratio between these two intervals is called a syntonic comma, defined as the interval that results adding four just fifths, minus two octaves, minus a just major third!

Pythagorean major thirds are "*peculiarly discordant and unsatisfactory*" and "*unsuitable for music in which major thirds and triads are treated as consonances*" but not inappropriate for "*a vast repertoire that spans the eighth through the fourteenth centuries*" when they were not treated as such.

If we keep moving further away from the starting point on the line of fifths we realize that we never go back to the original root note but we end up with an endless array of notes and that "*Pythagorean tuning, although generated by pure intervals, produces a variety of intervals that are out of tune*". So, for example, if in the standard equal temperament the size of twelve fifths equal that of seven octaves, in Pythagorean tuning they differ by a ratio appropriately named Pythagorean comma (see below X. J. Scott's definition of comma), that Blackwood calls interval (12) because we would find that interval twelve fifths away from the starting point on the line of fifths, once it is reduced within the compass of one octave!

Blackwood explains that, in order to get the largest possible number of consonant/ just intervals and triads, the seven notes of a Pythagorean diatonic scale are not enough and the only solution is to add further ones to be used depending on the role that a note has to play within an interval or triads. But these additional notes, separated by micro-intervals (commas), add harmonic consonance at the expense of melodic purity. He writes: "*the scale and the chords have independent requirements regarding their respective tunings, and an improvement in the one causes a deterioration in the other*".

Things get progressively more difficult if we want to use chords of four notes (as dominant seventh ones) instead of only three note chords (as major and minor ones). At this point of the book the reader is already surrounded by an almost inextricable maze of commas (diesis, schisma, diaschisma plus the previous ones) and other small intervals as limma, apoteme, major and minor chroma, maximum and diatonic semitone. Blackwood himself states: "*of course, there is no end to the number of commas formed by combinations of the first three basic intervals*". But things get even more complicated by the use of intervals derived from the fourth and the seventh basic intervals (7:4 and 17:16) generating even more commas, so much so that, at one point, the author says: "*thus far, our discussion of just tuning has brought about no fewer than eight different tunings for any note having one conventional name*". By the way, the fifth and sixth basic intervals (11:8 and 13:8) get discarded because they sound too "alien" for any diatonic scale.

All these efforts in the search of "*the most euphonious possible arrangement of all simultaneous combinations of notes*" bring Blackwood to the conclusion that "*a perfect tuning for a given composition cannot be found*" and he wittily add: "*it is hoped that the reader, as is the author, will be persuaded that the quest for perfection in tuning of a given musical fragment is the pursuit of an ignis fatuus* (Latin for a deceptive goal or hope)".

Blackwood's analysis moves to meantone tuning[3]: if a line of just fifths creates "*peculiarly discordant and unsatisfactory*" Pythagorean major thirds, we need to reduce the size of those fifths until major thirds sound just, if we want to treat these intervals as consonances. A Pythagorean major third is interval (4) in the line of fifths and because that interval is larger than pure by a syntonic comma (see above) we can reduce each of the four fifths by a quarter of the syntonic comma and the result will be a just major third. This is the classic quarter comma meantone tuning. There are also other variations always based on the principle to favor the consonance of thirds sacrificing that of fifths, as little as possible. It is like saying that if Pythagorean tuning system is based on ratios of prime numbers 2 and 3, meantone ones are based on ratios of prime numbers 2 and 5.

This process has its faults and benefits, Blackwood says: "*it appears that Pythagorean tuning favors the scale at the expense of the triads, whereas the opposite is true of meantone tuning*", nevertheless these two tuning systems "*sound like two different versions of the same thing*".

The author explains: "*there is abundant historical evidence that meantone tuning was the preferred tuning for keyboard instruments during the period 1550 to 1650*". Its great practical importance is that the slight impurity allowed by this tuning "*makes possible the elimination of the disturbing melodic discontinuities"* associated with previously mentioned commas.

The number of notes is limited to the twelve ones of the chromatic scale that usually go from Eb to G# (on the line of fifths) but only six of them can be tonics of a major key because for the remaining six "*one of the perfect fifths of the generating array is replaced by the diminished sixth G#Eb*".
It means that not all major and minor chords are usable, eight of the major ones and their relative minor chords are more euphonious than their analogous Pythagorean version, the other ones "*are called wolves, a quaint name suggested by a fancied similarity between their sound and that of a pack of wolves*"!

Clearly, "*the limitations of 12-note meantone tuning impose substantial restrictions on the style of keyboard compositions*", but it also produces an "*exotic charme*" which is lost when the same music is played in the familiar 12-note equal temperament.

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3 https://en.wikipedia.org/wiki/Meantone_temperament

Another famous "unequal temperament" mentioned by Blackwood is the one ascribed to Andreas Werckmeister[4] whose 12-note tuning gives "*something near meantone for the common triads, or Pythagorean tuning for the infrequent triads*", so, since it "*produces major thirds that vary from nearly pure to Pythagorean, it appears to be unsuited to styles in which all triads may be expected to occur with equal frequency*".

After all these historical analyses of tuning systems, Blackwood starts investigating "*the range over which the generating interval may vary to produce a diatonic tuning that exhibits the same subjective character and structural organization as Pythagorean and meantone tunings*" and, at least for me, things start getting more interesting.

The reader, to get here, has already gone through two thirds of the book (almost 200 pages). Personally, as a composer, I am not very interested in historical tuning systems, nevertheless Blackwood's approach to this matter is brilliant and very interesting. I skipped a few paragraphs because my limited mathematical skills would not allow me to confute or approve all his demonstrations, nevertheless I exactly understood what he meant because of my background as a microtonalist and "tuning explorer".

And now back to the review:
Blackwood demonstrates that as the size of the perfect fifth decreases from Pythagorean to meantone, the size of the other degrees of the diatonic scale change accordingly, following a pattern that depends on their position on the line of fifths!
For example, as the fifth decreases by one cent, the major third decreases by four cents because it is interval (4) on the line of fifths, while the fifth is interval (1). For this reason the size of all intervals of a diatonic major scale go flat when the fifth gets reduced in size, except the fourth, interval (-1) on the line of fifths. This demonstrates that "*an almost unnoticeable change in the perfect fifth produces a very perceptible change in the nature of the scale*".

The major second is interval (2) and the minor second is interval (-5) on the line of fifths, so, as the fifth decreases so does the major second, meanwhile the minor second increases its size. If the fifth gets progressively smaller "*there must come a point at which the major and minor seconds are equal*", making the scale not recognizable as a diatonic one because "*any two diatonic intervals named by the same ordinal number are equal*" (it means there is no difference between major and minor intervals anymore).

This point is reached when an octave is divided into seven equal steps and the fifth is 4/7 of an octave.

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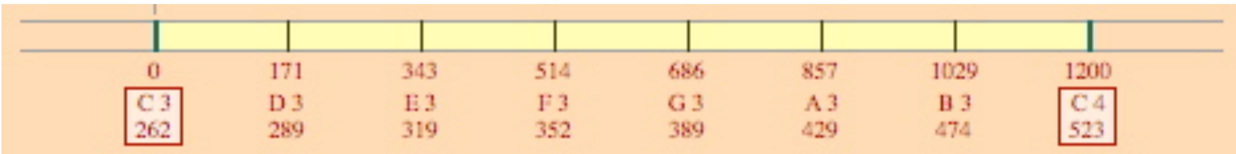
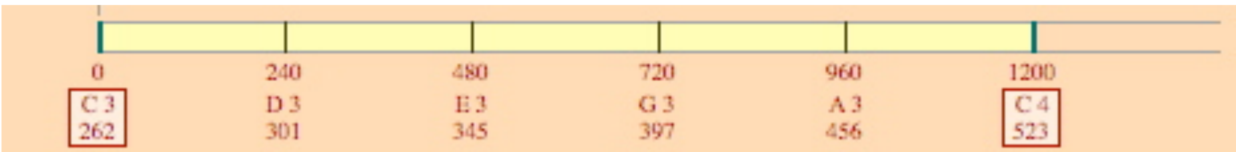
[4] https://en.wikipedia.org/wiki/Andreas_Werckmeister



Vice versa, if the size of the fifth increases, there must be a point when the minor second vanishes, again making the scale not recognizable as diatonic, and that point is reached when an octave is divided into five equal steps and the fifth is 3/5 of an octave.



Blackwood calls the range delimited by these two sizes of a fifth (4/7 and 3/5 of an octave[5]) the "*range of recognizability*", hence a "perfect fifth" is any interval lying between 4/7 and 3/5 of an octave, but not equal to them, that generates a diatonic scale.
He demonstrates that all principles applicable to historic diatonic tunings are also valid "*throughout the range of recognizability*".
This "*range of recognizability*" can also be expressed another way, in terms of major and minor seconds specifying that both be positive with the major one larger than the minor one, furthermore he introduces a value R as the ratio between major and minor second in order to define this "*range of recognizability*" in a neater way. For example, if R = 2 we have the standard equal division of an octave into 12 steps (12EDO), because the major second is twice the size of the minor one (of course, R = 2 also for all EDOs multiples of 12).

Ratio R gets larger as we move from meantone to Pythagorean tuning, "*with every note of the scale going sharp except the fourth degree, bringing the leading tone closer to the tonic. At the same time, the perfect fifths improve, going from slightly small to pure, while the major thirds deteriorate from pure to Pythagorean*".

A special class of recognizable diatonic tunings is the one in which perfect fifths form a closed circle. We have seen that neither Pythagorean nor meantone tunings form a closed circle of fifths. The Pythagorean one is more a spiral of fifths than a circle, never going back to the starting point and the meantone one is a broken circle of fifths with "*wolves*".

Blackwood demonstrates that "*a necessary and sufficent condition*" to produce a closed circle of fifths is that R should be a rational number (a number that can be written as a simple fraction) and Blackwood's conclusion is that "*it is not generally appreciated that the tuning where R = 2 is only one particular case of a general family of tunings*".

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[5] That is to say, 2^$^{(4/7)}$ and 2^$^{(3/5)}$, or 4 steps of the 7th root of 2/1, and 3 steps of the 5th root of 2/1

Here we enter the world of microtonality with examples of diatonic tunings where R is a rational number other than 2.
For R = 1.5 we have a "perfect fifth" 11/19 of an octave (divided into 19 equal steps) that is within the "*range of recognizability*".
For R = 3 we have a "perfect fifth" 10/17 of an octave (divided into 17 equal steps) that is also within the "*range of recognizability*".
Blackwood analyzes also equal tunings that are outside the "*range of recognizability*". I mention his Theorem 41 stating that EDOs containing recognizable diatonic scales are those of 12, 17, 19, 22, 24, 26, 27, 29, 31, 32, 33, 34, 36 or more notes!

In the end Blackwood examines "*the distribution and behavior of certain equal tunings that contain diatonic scales or approximations of just tunings*", that Blackwood calls "*diatonic equal tunings*".

So, for example, 12EDO can be seen as a temperament of Pythagorean tuning "*since the Pythagorean comma is distributed uniformly over twelve fifths*" and 31EDO "*is an extremely close approximation to extended meantone tuning*" because these two tunings have fifths almost the same size. He also takes into account other tunings having good approximations of just intervals like 34 EDO, 53EDO, 65EDO and up to 612 EDO.
22EDO too merits a scrutiny for the nearly just behavior of this tuning, due to its "*not impractically large number of notes*".
But Blackwood does not stop here, so, we come to know that "*there is no equal tuning of fewer than 665 notes that contain a closer approximation to a pure fifth than 389/665 of an octave*".
Blackwood's witty remark is: "*although the mathematical aspects of the subject are intriguing, the author cannot escape the conviction that they are of little interest musically*" and who am I to disagree with him?

A few thoughts I had while reading this book:

From my point of view, as microtonal composer, I feel there is something missing in this book. What would happen if instead of tempering fifths, leaving octaves unaltered, we would do the reverse? Would we find "quasi-diatonic equal tunings"? The answer is simple: YES!

I quote what X. J. Scott says about the Pythagorean comma in his "Glossary Of Musical Tuning Definitions[6]":

"*The difference between a stack of twelve fifths of 3/2 and a stack of seven octaves of 2/1 is* $(3:2)^{12} / (2:1)^{7}$*, which is 531441:524288, or 23.46 cents, roughly an eighth of a whole tone.*
Much ado is made about the fact that these two intervals don't match in a theoretical tuning of pure fifths called Pythagorean Tuning, and the western standard of twelve tone equal temperament is said to be the solution to this conflict.

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[6] http://www.nonoctave.com/tuning/glossary.html

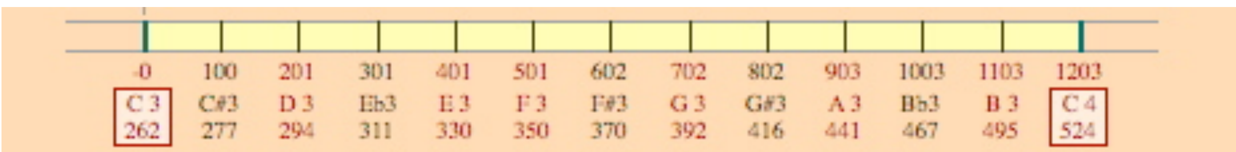
This conundrum, considered an unsolvable puzzle that has lasted centuries because of unchangeable facts of unyielding mathematics, is nothing more than a case of small-minded thinking inside of a box of one's own choosing.

First, there are an infinite number of useful and marvelous tunings that don't even need such constraints.

Second, if you really must have all pure fifths in a conventional sounding tuning, it is a trivial matter to use the seventh root of 3:2 as your basic chromatic step instead of the 12th root of 2:1.

This yields a slightly stretched octave of $(3:2)^{12/7}$ *or 1203.35 cents, which is not just only slightly and unnoticeably sharp of the 2/1 octave at 1200.0 cents, but it is basically the same octave that is used to tune all modern pianos anyway*[7]*.*

Desiccate the comma by tempering the octave instead of the fifth and you have a great and useful tuning that you are already familiar with."



This is such a brilliantly simple solution it is awesome!

This tuning, that X. J. Scott calls "Superpythagorean", was discovered by Serge Cordier[8].

I would also like to mention Wendy Carlos's Alpha, Beta and Gamma tunings, not repeating at the octave. Here what she says[9]: "*Several years ago I wrote a computer program to perform a precise deep-search investigation into this kind of Asymmetric Division, based on the target ratios of: 3/2, 5/4, 6/5, 7/4, and 11/8. Here's what it discovered.*

Between 10-40 equal steps per octave only three divisions exist which are amazingly more consonant than any other values around, like lush tropical islands scattered in a great ocean of uniform chaos. I call them Alpha ('alpha'), Beta ('beta'), and Gamma ('gamma'). These happy discoveries occur at:

'alpha' = 78.0 cents/step = 15.385 steps/octave,
'beta' = 63.8 cents/step = 18.809 steps/octave,
'gamma' = 35.1 cents/step = 34.188 steps/octave."

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7 https://en.wikipedia.org/wiki/Stretched_tuning

8 https://fr.wikipedia.org/wiki/Temp%C3%A9rament_%C3%A9gal_%C3%A0_quintes_justes

9 http://www.wendycarlos.com/resources/pitch.html

Each one of them has its own character:

"*The melodic motions of Alpha*[10] *are amazingly exotic and fresh, like you've never heard before.*"

"*Melodically it's quite impossible to hear much difference between Beta*[11] *and 19-tone Equal. So Beta is suited for more standard types of music which might benefit from the nearly perfect harmonies*."

"*Gamma*[12] *... is also slightly smoother than Alpha or Beta, having no palpable difference from Just tuning in harmonies, which is saying a lot. You really have to go further, up to 53-step E.T., to find another nearly perfect equal division, yet Gamma is noticeably freer of beats than even that venerable tuning.*"

In conclusion, I am glad I read this book and spent time studying it because Blackwood is a great composer (listen to his Twelve Microtonal Etudes For Electronic Media) and his solid approach to this matter is definitive.

By the way, what Blackwood[13] writes about his Microtonal Etudes is very interesting: "*Why should one compose in equal tunings of 13 through 24 notes*[14] *when some of these tunings sound decidedly out of tune and have previously been regarded as of little or no musical utility? Long before I began the Microtonal Etudes project, I doubted the wisdom of theorists who rejected out of hand any tuning that did not produce consonant triads. Only when I began this study, however, did I discover the vast musical possibilities inherent in different equal tunings. I believe these Etudes prove that equal tunings of more than twelve notes can indeed produce expressively compelling progressions of hitherto alien harmonies and modes.*"

Clearly, there are other ways of exploring tuning systems outside the realm of diatonic ones, so, it is time I go back composing!

(Thanks to X. J. Scott for proofreading this paper)

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[10] http://xenharmonic.wikispaces.com/Carlos+Alpha

[11] http://xenharmonic.wikispaces.com/Carlos+Beta

[12] http://xenharmonic.wikispaces.com/Carlos+Gamma

[13] http://www.dramonline.org/albums/blackwood-microtonal-compositions/notes

[14] per octave